2-LC triangulated manifolds are exponentially many

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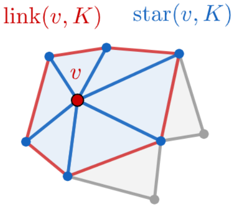
Background picture

- *Facets* are inclusion-maximal faces of a regular CW-complex.
- *Pure*: a regular CW-complex with facets of same dimension.
- *Star* of a face σ: the smallest subcomplex containing all facets that contain σ.



- $\mathrm{link}(\sigma, K) = \{\tau \in \mathrm{star}(\sigma, K) : \tau \cap \sigma = \emptyset\}$ if K is a simplicial complex. If K is a regular CW-complex: "spherical link"
- *Triangulation of a smooth d-manifold M*: a d-dim simplicial complex whose underlying space is homeomorphic to M.
- *d-sphere*: A triangulation of the d-dimensional sphere.
- *d-pseudomanifold*: a d-dim pure simplicial regular CW-complex where each $(d-1)$-cell is in ≤ 2 facets.

Big picture

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Steinitz's theorem

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Due to discrete quantum gravity (which cheers for 'yes' – cf. Riccardo Martini's talk yesterday) we count triangulations with respect to the number N of facets.

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How many 3-spheres with N tetrahedra, exponentially many?

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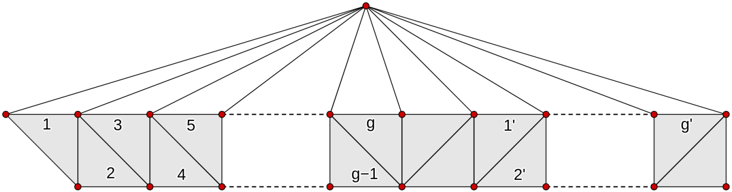
Theorem [Folklore]

There are $g!$ triangulations of the genus-g surface, with $16g$ triangles each. Hence, there are more than exponentially many surfaces with N triangles.



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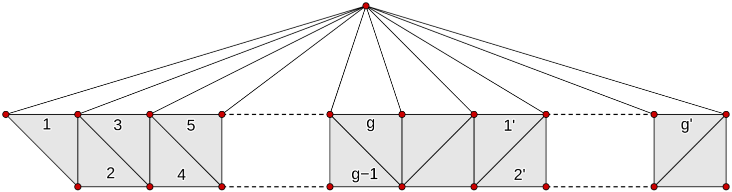
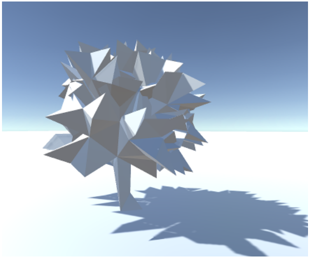


Corollary (via coning)

There are more than exponentially many 3-pseudomanifolds with N tetrahedra.

A picture dear to me: LC Manifolds

- LC manifolds are those obtainable from a tree of d-simplices by recursively gluing two *adjacent* boundary facets.
- Mogami manifolds: ... by gluing *incident* boundary facets.
- All shellable (or constructible) spheres are LC.



Theorem (Durhuus–Jonsson 1995; B.–Ziegler 2011)

LC triangulations of d-manifolds with N facets are at most $2^{d^2 N}$.

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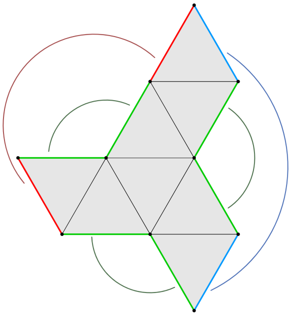

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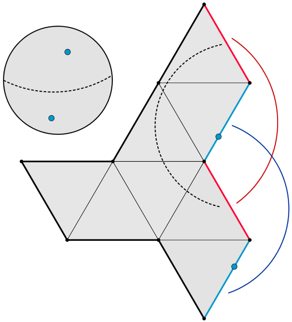

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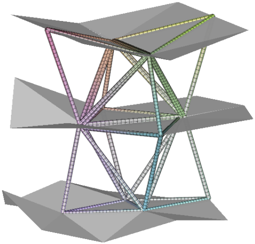
Later pictures

Classes with proven exponential size:

- (d=2) Surfaces with fixed genus (Tutte 1962)
- (d=3) Causal triangulations (Durhuus–Jonsson 2014)
- (any d) Bounded geometry, á-la-Cheeger (Adiprasito–B. 2020)



- Triangulations with bounded discrete Morse vector – contains all classes above, and also LC triangulations, but not Mogami triangulations (Benedetti 2012)

B.–Ziegler (2011)

A d-manifold without boundary is LC if and only if it admits a discrete Morse vector that ends in $\dots 0, 1)$.

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Let t be an integer ranging from 1 to d.

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Definition (B.–Pavelka 2023)

t-LC d-manifolds are those obtainable from a tree of d-simplices by recursively gluing two boundary facets whose intersection has dimension at least $d - 1 - t$.

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- 1-LC $\subset$ 2-LC $\subset \cdots \subset$ d-LC
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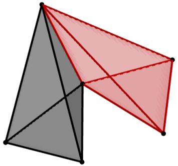

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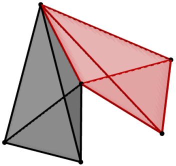

Main theorem (B.–Pavelka 2023)

2-LC triangulations of d-manifolds with N facets are still exponentially many: they are at most $2^{\frac{d^3}{2}N}$.

Special effects: Coning

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Cones over t-LC d-pseudomanifolds are t-LC.

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Crucial facts for our proof

- Links of $(d-3)$-faces in a manifold are homeomorphic to S^2 or a disk.
- Planar gluings lead to count by Catalan numbers.
- Our proof makes precise and extends to all dimension the work for $d = 3$ by Mogami.

Another famous picture: Cohen-Macaulayness

- A d-dimensional complex C is called *[homotopy-]Cohen–Macaulay* if for any face F, for all $i < \dim \operatorname{link}(F, C)$, [$\pi_i(\operatorname{link}(F, C)) = 0$ and] $H_i(\operatorname{link}(F, C)) = 0$.

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- *Constructible simplicial complex* defined inductively:
  - every simplex, and every 0-complex, is constructible;
  - a d-dim pure simplicial complex C that is not a simplex is constructible if and only if it can be written as $C = C_1 \cup C_2$, where C_1 and C_2 are constructible d-complexes, and $C_1 \cap C_2$ is a pure constructible $(d-1)$-complex.

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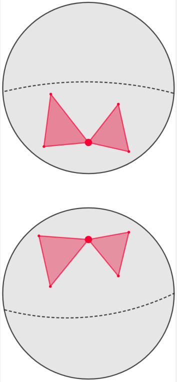
- Constructible complexes are homotopy-Cohen–Macaulay (Hochster 1972).
- Constructible manifolds are LC. (B.–Ziegler 2011)

... And here is another movie!

Definition (B.–Pavelka 2023)

Let $0 < t \leq d$ be integers. *t-constructible* d-dimensional simplicial complexes defined inductively:

- every simplex is t-constructible;
- a 1-dimensional complex is t-constructible if connected;
- a d-dimensional pure simplicial complex C that is not a simplex is t-constructible if $C = C_1 \cup C_2$, where C_1 and C_2 are t-constructible d-complexes, and $C_1 \cap C_2$ is a pure $(d-1)$-complex whose $(d-t)$-skeleton is constructible.



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