Spin structures on Pseudo-Riemannian Cobordisms

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Coexistence Issues of Spin structures and Pseudo-Riemannian Metrics on Cobordisms

Definition

A **cobordism** is a triple $(W; M_1, M_2)$ where W is a smooth compact $(n+1)$-manifold with non-empty boundary $\partial W = M_1 \sqcup M_2$.

Question

What topological conditions should be imposed on M_1 and M_2 in order to admit a cobordism $(\hat{W}; M_1, M_2)$ which is both Spin and has a non-singular pseudo-Riemannian metric of signature $(2, n-1)$ restricting to a Lorentzian metric on its boundary?

The 3-dimensional case

Let M_1, M_2 be closed orientable (smooth) 3-manifolds.

- There exists a Spin-cobordism

$$(W; M_1, M_2)$$

  (by Milnor's computation of Spin-cobordism groups).

- There exists a cobordism

$$((\hat{W}; M_1, M_2), g)$$

  where g is a Lorentzian metric on $\hat{W}$ restricting to a Riemannian metric on its boundary (by a result of Reinhart [3] and Sorkin [5]).

Problem

The manifold $\hat{W}$ in the cobordism above need not support a Spin structure!

A result by Gibbons-Hawking

Theorem (Gibbons-Hawking [2])

Let $\{M_1, M_2\}$ be closed oriented 3-manifolds. The following conditions are equivalent

1. *There exists a Spin-cobordism $(W; M_1, M_2)$ admitting a Lorentzian metric which restricts to a Riemannian one on its boundary $\partial W = M_1 \sqcup M_2$*
2. $\hat{\chi}_{\mathbb{Z}/2}(M_1) = \hat{\chi}_{\mathbb{Z}/2}(M_2)$
3. *W is parallelizable and $\chi(W) = 0$.*

The topological invariant

$$\hat{\chi}_{\mathbb{Z}/2}(X) := \sum_{i=0}^{k} b_i(X; \mathbb{Z}/2) \mod 2 \tag{1}$$

is called **Kervaire semi-characteristic** and it can be defined for any closed $(2k+1)$-manifold X. Here $b_i(X; \mathbb{Z}/2)$ denotes the i^{th} Betti number of X with $\mathbb{Z}/2$-coefficients.

Pseudo-Riemannian metrics on Spin-cobordisms

Let M_1, M_2 be closed smooth Spin n-manifolds.

Definition

A $\mathrm{Spin}_0(1,n)$-Lorentzian cobordisms is a pair $((W; M_1, M_2), g)$ where:

1. $((W; M_1, M_2), g)$ is a Spin-cobordism;
2. $(W; g)$ is a Lorentzian metric restricting to a Riemannian one on $\partial W = M_1 \sqcup M_2$.

- Such cobordisms have been studied by Smirnov and Torres in [4].
- The topic of Lorentzian cobordism has been addressed by many mathematicians and mathematical physicists, including Chamblin, Gibbons-Hawking [2], Reinhart [3], Sorkin [5].
- Apart from some partial results by Alty and Chamblin [1], not much is known about other kinds of Spin pseudo-Riemannian cobordisms.

The main definition

Definition

Let M_1, M_2 be closed smooth Spin n-manifolds. A $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordism between M_1 and M_2 is a pair

$$((W; M_1, M_2), g)$$

that consists of:

- a Spin-cobordism $(W; M_1, M_2)$;
- a non-singular indefinite metric (W, g) of signature $(2, n-1)$ such that
- the restriction of g to the boundary $\partial W = M_1 \sqcup M_2$ gives rise to Lorentzian metrics, i.e. nonsingular indefinite metrics of signature $(1, n-1)$.

Spin$(2,2)_0$-pseudo-Riemannian cobordisms

Theorem (B. - May Custodio - Torres, cfr. Alty-Chamblin [1])

Let $\{M_1, M_2\}$ be closed oriented 3-manifolds. The following conditions are equivalent

1. *There exists a Spin$(2,2)_0$-pseudo-Riemannian cobordism*
$$((W; M_1, M_2), g),$$
2. $\hat{\chi}_{\mathbb{Z}/2}(M_1) = \hat{\chi}_{\mathbb{Z}/2}(M_2)$
3. *W is parallelizable and $\chi(W) \equiv 0 \mod 2$.*

There is a group isomorphism

$$\Omega^{Spin_0}_{2,2} \to \mathbb{Z}/2. \tag{2}$$

A key example

- The product of a 2-disk with the 2-sphere admits a Spin-structure as well as an indefinite metric of signature $(2,2)$
$$(D^2, -dr^2 - r^2 d\theta^2) \times (S^2, g_{S^2}) \tag{3}$$
that restrict to a Spin-structure and a Lorentzian metric on the boundary
$$(S^1, -d\theta^2) \times (S^2, g_{S^2}). \tag{4}$$
- (r, θ) are polar coordinates on the 2-disk, while g_{S^2} is the usual round metric on the 2-sphere.
- The Euler characteristic of (3) is $\chi(D^2 \times S^2) = 2$ and it does not admit a Lorentzian metric that restricts to a Riemannian metric on (4). The connected sum
$$\hat{W} = D^2 \times S^2 \# S^1 \times S^3,$$
does support both kinds of non-degenerate pseudo-Riemannian metrics as well as Spin-structures that restrict to (4).

Our Main Result

Theorem (B. - May Custodio - Torres)

Let $\{M_1, M_2\}$ be closed smooth Spin-cobordant n-manifolds.

- *Suppose* $n \not\equiv 1, 5, 7 \mod 8$. *There exists a* $Spin(2, n-1)_0$*-pseudo-Riemannian cobordism* $((W; M_1, M_2), g)$ *if and only if*

  1. $\chi(M_1) = \chi(M_2) = 0$ *for n even*
  2. $\hat{\chi}_{\mathbb{Z}_2}(M_1) = \hat{\chi}_{\mathbb{Z}_2}(M_2)$ *for n odd.*

- *Suppose* $n \equiv 1 \mod 4$. *If* $\hat{\chi}_{\mathbb{Z}_2}(M_1) = \hat{\chi}_{\mathbb{Z}_2}(M_2)$, *there exists a* $Spin(2, n-1)_0$*-pseudo-Riemannian cobordism.*

- *Suppose* $n \equiv 7 \mod 8$. *There is a* $Spin(2, n-1)_0$*-pseudo-Riemannian cobordism without any further assumptions.*

2-distributions

Main tool:

Results by Emery Thomas [6] on sufficient and necessary conditions for the existence of distributions of tangent 2-planes on a given n-manifold X.

Indeed, given a closed smooth n-manifold X, the following are equivalent:

- there exists a pseudo-Riemannian metric (X, g) of signature $(2, n-2)$;
- there exists a rank 2 sub-bundle $\xi \subset TX$.

Results in low dimensions

Theorem (B. - May Custodio - Torres)

Let $\{M_1, M_2\}$ be closed Spin n-manifolds.

• *If $n = 4$, there is a $Spin(2,3)_0$-pseudo-Riemannian cobordism $((W; M_1, M_2), g)$ if and only if $\chi(M_1) = 0 = \chi(M_2)$ and $\sigma(M_1) = \sigma(M_2)$. There is a group isomorphism*

$$\Omega_{2,3}^{Spin_0} \to \mathbb{Z}. \tag{5}$$

• *If $n = 6$, there is a $Spin(2,5)_0$-pseudo-Riemannian cobordism $((W; M_1, M_2), g)$ if and only if $\chi(M_1) = 0 = \chi(M_2)$.*

The group $\Omega_{2,5}^{Spin_0}$ is trivial.

• *If $n = 7$, there is a $Spin(2,6)_0$-pseudo-Riemannian cobordism $((W; M_1, M_2), g)$ without any further assumptions.*

The group $\Omega_{2,6}^{Spin_0}$ is trivial.

References

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Thank you for the attention :)