A tour into Dunwoody manifolds

Alessia Cattabriga
(University of Bologna)

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When everything (at least for me) started...

Forum Math. **13** (2001), 379–397

Forum Mathematicum

Genus one 1-bridge knots and Dunwoody manifolds*

Luigi Grasselli and Michele Mulazzani

(Communicated by Karl Strambach)

Abstract. In this paper we show that all 3-manifolds of a family introduced by M. J. Dunwoody are cyclic coverings of lens spaces (possibly $\mathbf{S}^3$), branched over genus one 1-bridge knots. As a consequence, we give a positive answer to the Dunwoody conjecture that all the elements of a wide subclass are cyclic coverings of $\mathbf{S}^3$ branched over a knot. Moreover, we show that all branched cyclic coverings of a 2-bridge knot belong to this subclass; this implies that the fundamental group of each branched cyclic covering of a 2-bridge knot admits a geometric cyclic presentation.

1991 Mathematics Subject Classification: 57M12, 57M25; 20F05, 57M05.

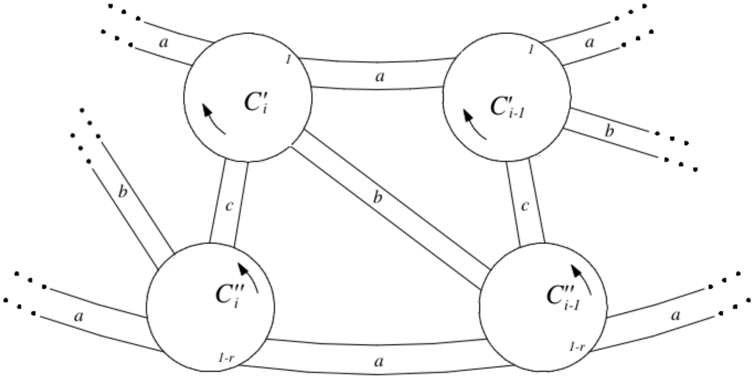
Plan of the talk

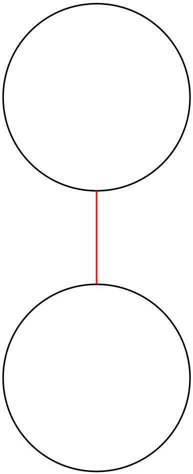
1. Dunwoody manifolds.
2. Characterization via branched coverings.
3. Complexity estimation.
4. Link equivalence in Dunwoody manifolds.
5. Future works.

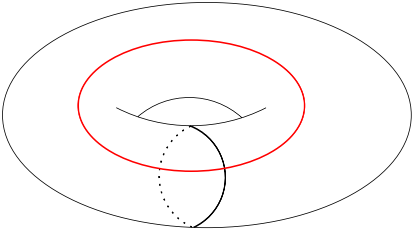
1. Dunwoody manifolds: definition

Definition (Dunwoody 1995)

Given the 6-tuple of integers (a, b, c, n, r, s), the Dunwoody manifold $M(a,b,c,n,r,s)$ is the **closed** connected orientable 3-manifold having the graph $\Gamma(a,b,c,n)$, with the circle C_i identified with C_{i+s} mod n and respecting vertices' labelling, as (open) Heegaard diagram, for $n > 0$, $a, b, c \geq 0$ and (a, b, c, n, r, s) admissible.

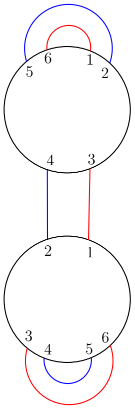




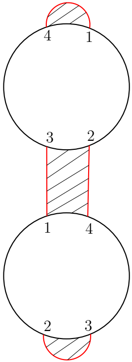


$$M(0,0,1,1,0,0) \cong S^3$$

Two examples of 6-tuples that are NOT admissible.



$M(2,0,2,1,2,0)$



$M(1,0,2,1,2,0)$

1. Dunwoody manifolds: examples

- $n = 1$ lens spaces (i.e. cyclic quotients of S^3), S^3, $S^2 \times S^1$
- $M(1, 0, 0, n, 1, 0)$ connected sum of n copies of $S^2 \times S^1$
- Minkus manifolds (Fibonacci manifolds and fractional Fibonacci manifolds)
- Sieradsky manifolds
- periodic Takahashi manifolds $T_n(1/q, 1/r)$
- Seifert manifolds[1] $(Oo, 0 \mid -1; \underbrace{(p, q), \ldots, (p, q)}_{n-\text{times}}, (l, l-1))$.

[1] Grasselli L. and Mulazzani M., Seifert manifolds and $(1,1)$-knots, *Siberian Math. J.* **50** (2009), 22-31.

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$$G_n(w) = \langle x_1, \ldots, x_n \mid \theta^j(w),\ j = 0, \ldots n-1 \rangle$$

with $\theta(x_i) = x_{i+1}$ subscripts mod n.

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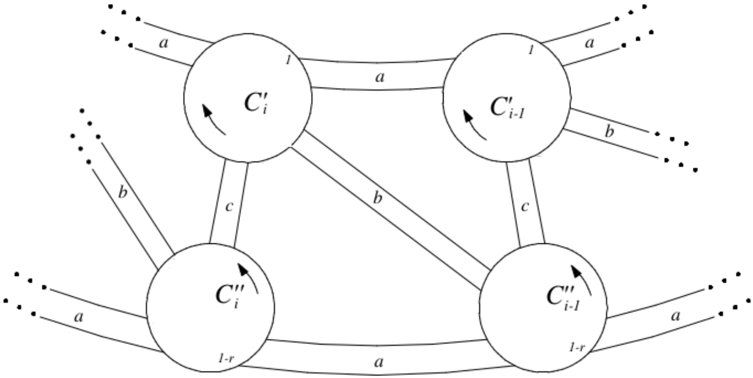
Howie Williams 2020: Study more general graphs with cyclic symmetry/cyclically presented groups.

2. Dunwoody manifolds: characterisation



Theorem (Grasselli, Mulazzani 2001)

[a] *Every Dunwoody manifold is a n-fold strongly-cyclic covering of a lens space (possibly S^3 or $S^2 \times S^1$) branched over a $(1,1)$-knot.*

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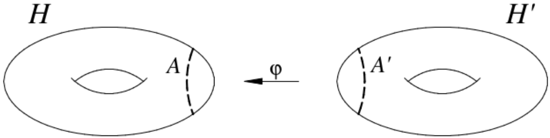
Theorem (C., Mulazzani 2003)

Every n-fold strongly-cyclic covering of a lens space (possibly S^3 or $S^2 \times S^1$) branched over a $(1,1)$-knot is a Dunwoody manifold.

The branching set is a (1,1)-knot (or genus one 1-bridge knot).



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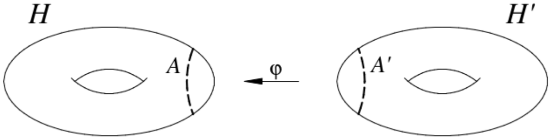
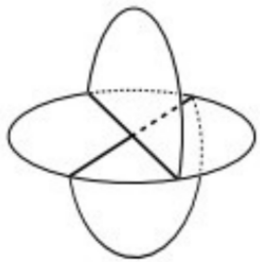


The covering is strongly-cyclic, that it is cyclic and the branching index at each point of the branching set equals the number of sheets of the covering.

3. Dunwoody manifolds: complexity estimation

Definition (Matveev 1990)

The complexity $c(M)$ of a closed 3-manifold M is the minimum number of true vertices among all almost simple spines of M.



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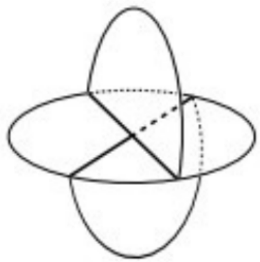

An almost simple spines of M (closed) is an embedded polyhedron P such that:

- $\left(M - \text{int}(B^3)\right) \searrow P$
- the link of each point in P can be embedded into K_4.

A true vertex of an almost simple spine is a point whose link is K_4.

3. Dunwoody manifolds: complexity estimation

Proposition (Matveev 1990)

For closed irreducible and $\mathbb{P}^2$- irreducible manifolds the complexity coincide with **the minimum number of tetrahedra** needed to construct a manifold (by gluing tetrahedra along their faces with affine identifications), with the only exceptions of S^3, $\mathbb{RP}^3$ and $L(3,1)$, all having complexity zero.

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Compute complexity is, in general a very difficult task!

3. Dunwoody manifolds: complexity estimation

Proposition (C., Mulazzani, Vesnin 2010)

Let $M = M(a,b,c,n,r,s)$ be a Dunwoody manifold.

(i) If $abc > 0$ then

$$c(M) \leqslant \begin{cases} n(2a+b+c) - \max(2n,6) & \text{if } r \neq -b, -b \pm 1, \\ n(2a+b+c) - \max(2n,5) & \text{if } r = -b \pm 1. \end{cases}$$

(ii) If $abc = 0$ and $\min(a, b+c) = 0$ then

$$c(M) \leqslant \begin{cases} n(2a+b+c-4) & \text{if } r \neq -b, -b \pm 1, \\ n(2a+b+c-3) & \text{if } r = -b \pm 1. \end{cases}$$

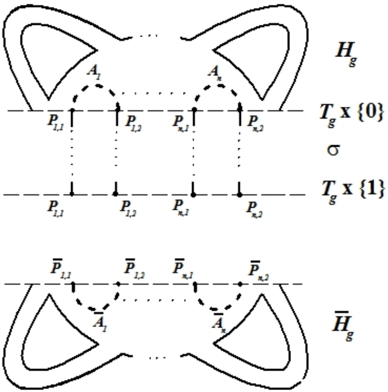
(iii) If $abc = 0$ and $\min(a, b+c) > 0$ then

$$c(M) \leqslant \begin{cases} n(2a+b+c-2) & \text{if } n > 3, \\ n(2a+c) - \max(2n, 8-2k_0) & \text{if } n = 2,3,\ b = 0 \text{ and } s = 0, \\ n(2a+b) - \max(2n, 8-k_0-k_1) & \text{if } n = 2, c = 0 \text{ and } s = 0, \\ n(2a+b) - \max(2n, 8-k_0) & \text{if } n = 3, c = 0 \text{ and } s = 0, \\ n(2a+b) - \max(2n, 8-k_1) & \text{if } n = 3, c = 0 \text{ and } s = 1, \end{cases}$$

where $k_i = \begin{cases} 2 & \text{if } r = (-1)^i b, \\ 1 & \text{if } r = (-1)^i b \pm 1, \\ 0 & \text{otherwise.} \end{cases}$

4. Link equivalence in Dunwoody manifolds

M closed connected orientable 3-manifold and T_g a **Heegaard surface** for M.



Theorem (Birman 1976, Doll 1992, C.-Mulazzani 2008, Bellingeri-C. 2012)

Every link in M is (isotopic to) the plat closure $\widehat{\sigma}$ of an element σ in the surface braid group $B_{2n}(T_g)$.

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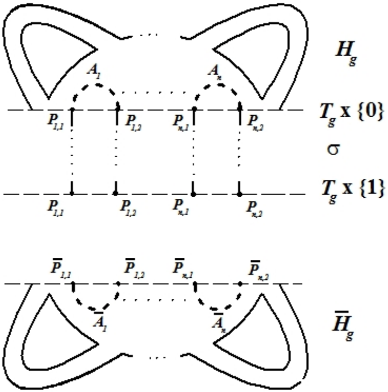

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Theorem (C.-Gabrovšek 2018)

Two links $\widehat{\beta_1}$ and $\widehat{\beta_2}$ in M are isotopic iff $\beta_1 = \beta_2$ in $B_{2n}(T_g)$ up to

- *multiplication by "Hilden" type elements $\rightsquigarrow$ isotopy in the thickened surface*
- *stabilization*
- *multiplication by braid representative of meridian curves $\rightsquigarrow$ isotopy through the handles.*

4. Link equivalence in Dunwoody manifolds

$$B_{2n}(T_g) = \langle \sigma_1, \ldots, \sigma_{2n-1}, a_1, \ldots, a_g, b_1, \ldots, b_g \mid \text{relations} \rangle$$

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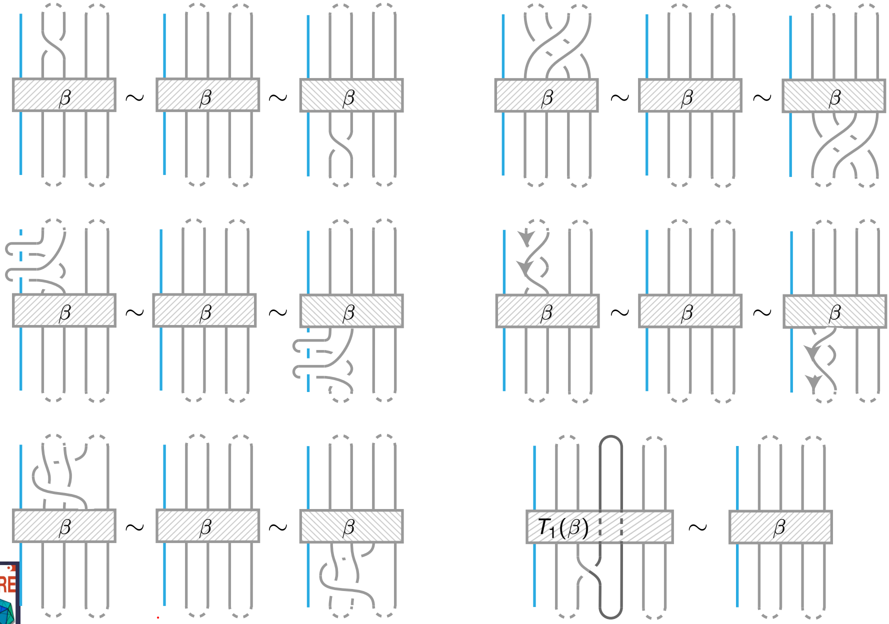
$$
\begin{array}{rl}
(M1) & \sigma_1\beta \longleftrightarrow \beta \longleftrightarrow \beta\sigma_1 \\
(M2) & \sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i}\beta \longleftrightarrow \beta \longleftrightarrow \beta\sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i} \; i = 1, \dots, n \\
(M3) & \sigma_2\sigma_1^2\sigma_2\beta \longleftrightarrow \beta \longleftrightarrow \beta\sigma_2\sigma_1^2\sigma_2 \\
(M4) & a_j\sigma_1^{-1}a_j\sigma_1^{-1}\beta \longleftrightarrow \beta \longleftrightarrow \beta a_j\sigma_1^{-1}a_j\sigma_1^{-1} \quad \text{for } j = 1, \dots, g \\
(M5) & b_j\sigma_1^{-1}b_j\sigma_1^{-1}\beta \longleftrightarrow \beta \longleftrightarrow \beta b_j\sigma_1^{-1}b_j\sigma_1^{-1} \quad \text{for } j = 1, \dots, g \\
(S) & \beta \longleftrightarrow T_k(\beta)\sigma_{2k}
\end{array}
$$

where $T_k : B_{g,2n} \to B_{g,2n+2}$ is defined by $T_k(a_i) = a_i, \; T_k(b_i) = b_i$,

$$T_k(\sigma_i) = \begin{cases} \sigma_i & \text{if } i < 2k \\ \sigma_{2k}\sigma_{2k+1}\sigma_{2k+2}\sigma_{2k+1}^{-1}\sigma_{2k}^{-1} & \text{if } i = 2k \\ \sigma_{i+2} & \text{if } i > 2k \end{cases}.$$

$$(psl_i/plsl_i*) \qquad w_j\beta \longleftrightarrow \beta \longleftrightarrow \beta b_j \quad \text{for } j = 1, \dots, g$$

where $w_1, \dots, w_g$ are words that represent the non standard meridian curves of the splitting.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4. Link equivalence in Dunwoody manifolds

Theorem (C., Cavicchioli 2023)

If $M = M(a, b, c, n, r, s)$ is a Dunwoody manifold the words $w_1, \dots, w_n$ can be algorithmically computed from (a, b, c, n, r, s) with complexity $O(N)$ with $N = (2a + b + c)n$.

5. Future works

- Determine explicit expression for w_j for some families of Dunwoody manifolds.

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- Understand the dependence on the Heegaard surface: study the problem in Seifert that are Dunwoody.

Thanks for your attention