Protocorks and exotic 4-manifolds

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June 6, 2023
Geometric Topology, Arts and Science

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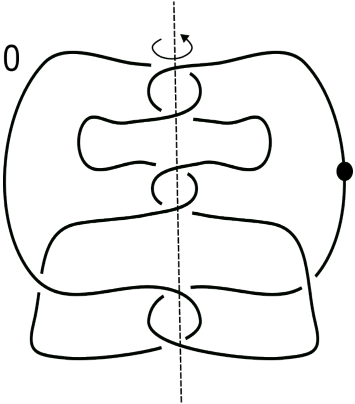

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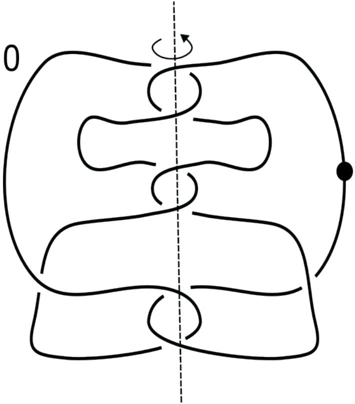

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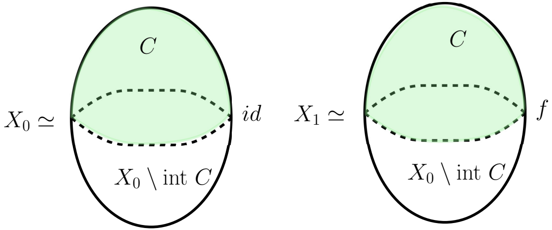

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- the action of a cork depends on the action of a *supporting protocork*.
- In contrast with corks we can enumerate them.

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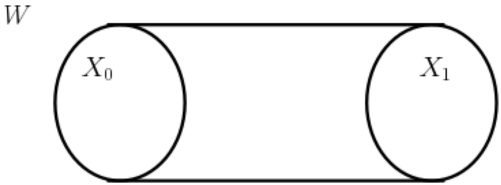
(X_0, X_1) exotic, $(\pi_1 X_0 = 1)$, then exists $W : X_0 \to X_1$ h-cobordism



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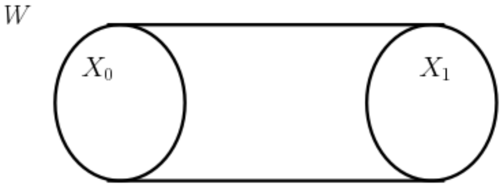

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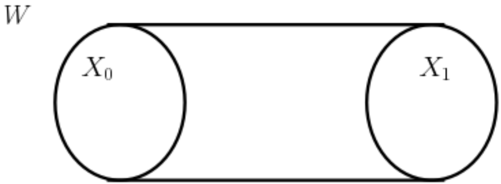

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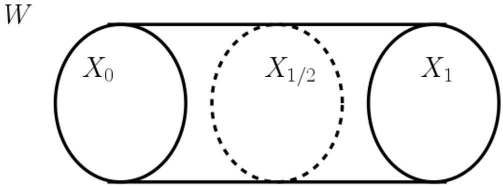

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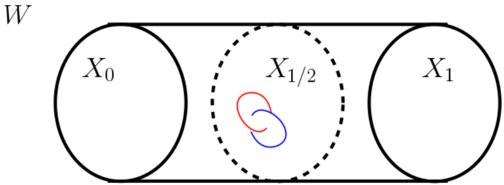

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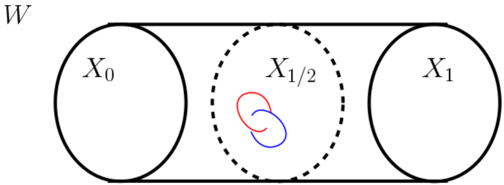

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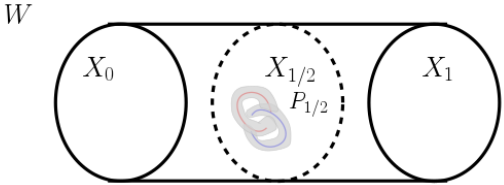

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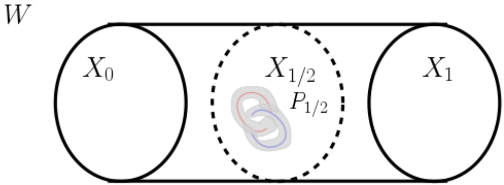

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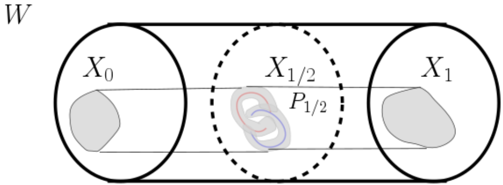

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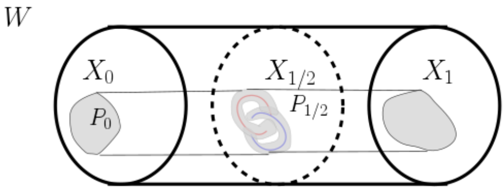

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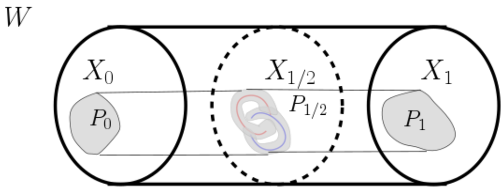

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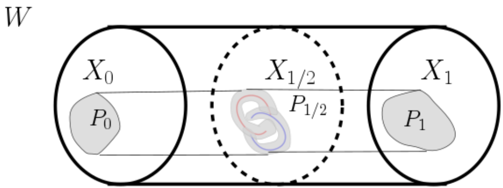

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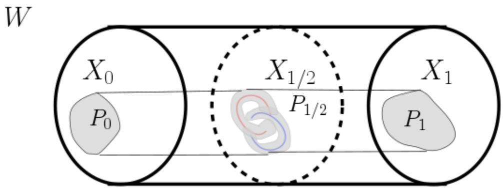

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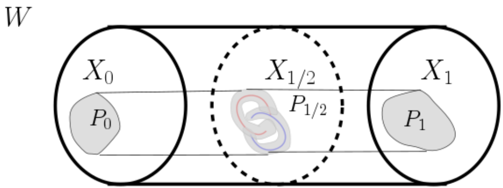

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- $X_0 \simeq M \bigcup P_0$, $X_1 \simeq M \bigcup_\alpha P_1$ "protocork twist".

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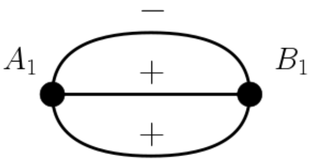

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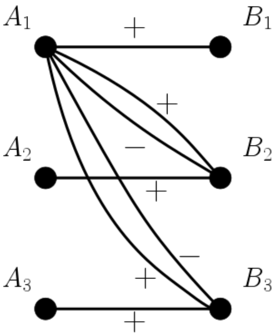

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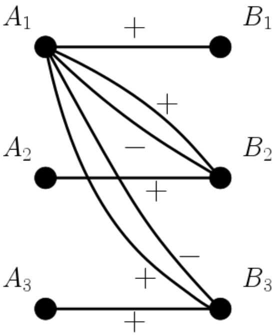

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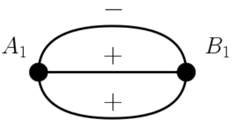
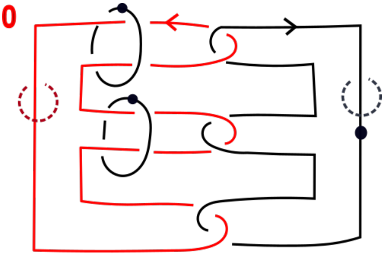
$P_0($ 

$) \simeq$ 

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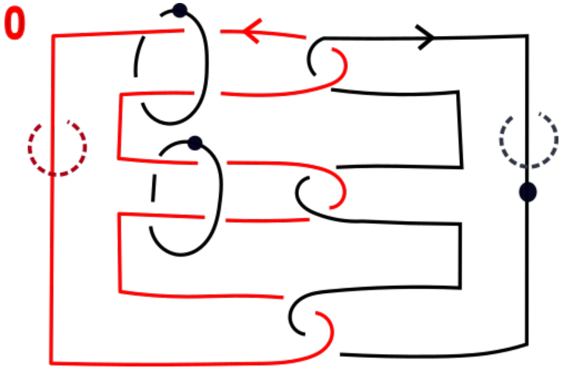

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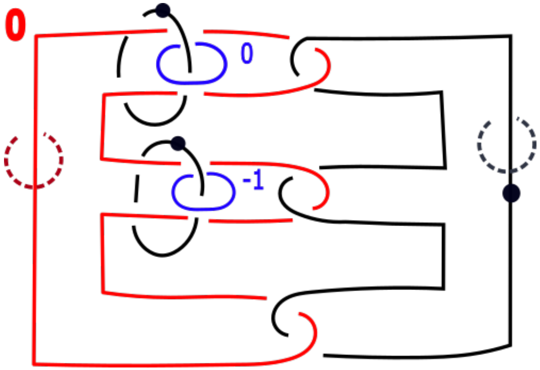

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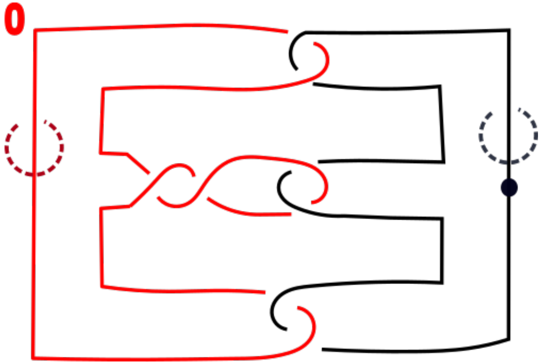

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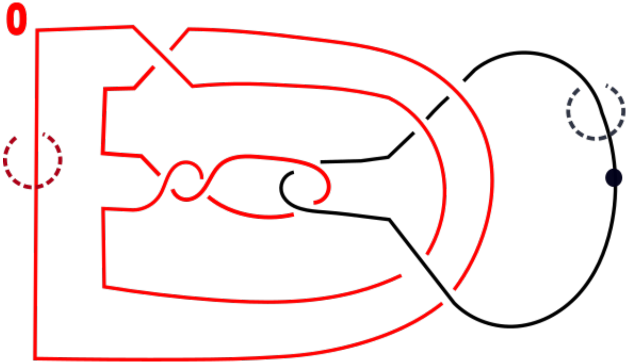

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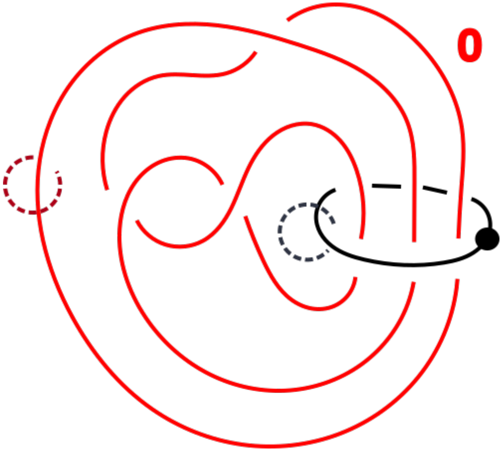

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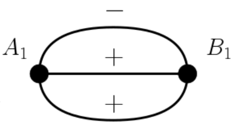

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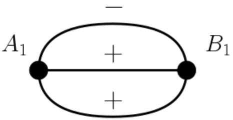
$P_0($ 

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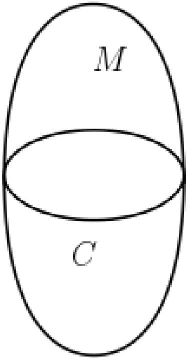
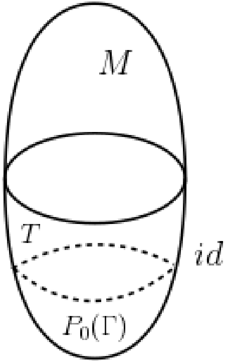
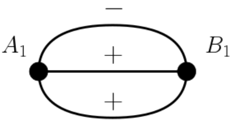
$X_0 =$ 

$=$ 

$P_0($ 

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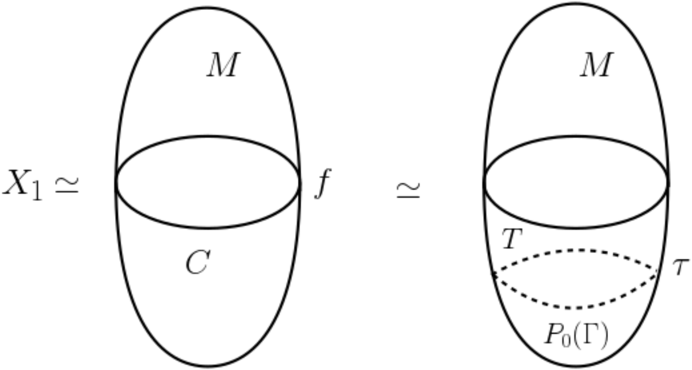
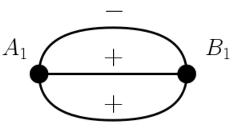


$P_0($ 

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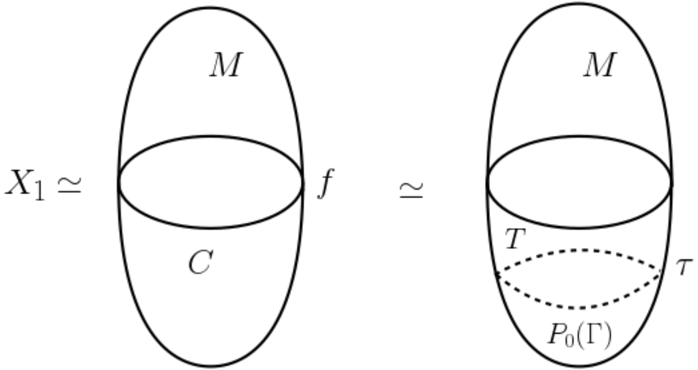

Prop. Any cork (C, f) admits a supporting protocork.

- Problem: does a protocork twist change the Seiberg-Witten invariants?

To analyse this we use Monopole Floer homology (Kronheimer-Mrowka '07). Recap:

- To Y^3 oriented, closed 3-manifold we associate $\mathbb{Z}[[U]]$-modules

$$\widetilde{HM}_\bullet(Y), \quad \widehat{HM}_\bullet(Y), \quad \overline{HM}_\bullet(Y)$$

- Ex. $\widehat{HM}_\bullet(\mathbb{S}^3) \simeq \mathbb{Z}[[U]]$, $\widetilde{HM}_\bullet(\mathbb{S}^3) \simeq \frac{\mathbb{Z}[U^{-1}, U]]}{U \cdot \mathbb{Z}[[U]]}$,
- W^4 cobordism $W : Y_0 \to Y_1$, induces

$$\widehat{HM}_\bullet(W) : \widehat{HM}_\bullet(Y_0) \to \widehat{HM}_\bullet(Y_1)$$

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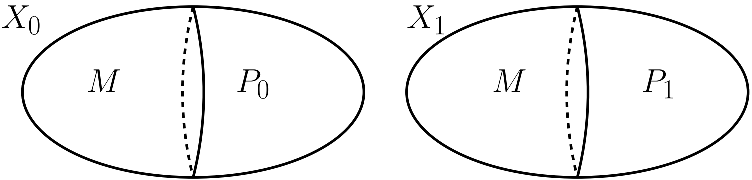

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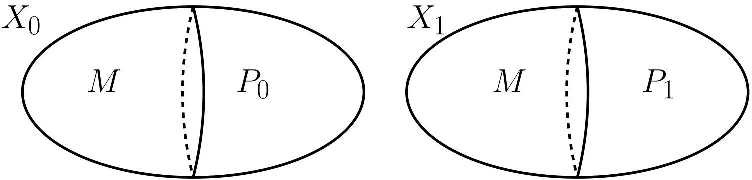

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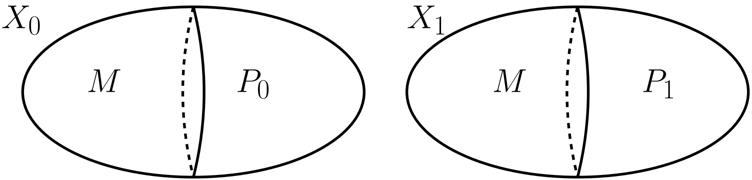

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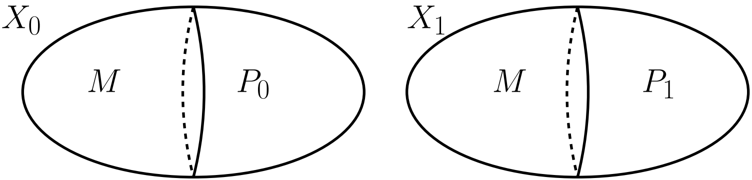

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Kronheimer-Mrowka:

$$SW(X_0, \mathfrak{s}_0) - SW(X_1, \mathfrak{s}_1) = \mathcal{L}_{M, \mathfrak{s}_M}(\Delta) \mod 2$$

where $\mathcal{L}_{M, \mathfrak{s}_M}$ is $\mathbb{Z}$-homomorphisms.

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- Hence $(\overline{C}, f)$ cannot change SW-invariants mod 2,

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