L-spaces, taut foliations and fibered hyperbolic two-bridge links

Diego Santoro

09 June 2023

Scuola Normale Superiore, Pisa

Plan of the talk

(I) Introduction to the L-space conjecture

(II) The conjecture and two-bridge links

(III) Application to Whitehead doubles

The L-space conjecture.

Annals of Mathematics, **159** (2004), 1027–1158

Holomorphic disks and topological invariants for closed three-manifolds

By PETER OZSVÁTH and ZOLTÁN SZABÓ*

Abstract

The aim of this article is to introduce certain topological invariants for closed, oriented three-manifolds Y, equipped with a Spin^c structure. Given a Heegaard splitting of $Y = U_0 \cup_\Sigma U_1$, these theories are variants of the Lagrangian Floer homology for the g-fold symmetric product of Σ relative to certain totally real subspaces associated to U_0 and U_1.

M^3 closed oriented $\longrightarrow \widehat{HF}(M)$, finite dim. $\mathbb{F}_2$-vector space

L-spaces

Fact: For a rational homology sphere M, $\dim \widehat{HF}(M) \geq |H_1(M, \mathbb{Z})|$.

L-spaces

Fact: For a rational homology sphere M, $\dim \widehat{HF}(M) \geq |H_1(M, \mathbb{Z})|$.

Definition: M is an L-space if $\dim \widehat{HF}(M) = |H_1(M, \mathbb{Z})|$.

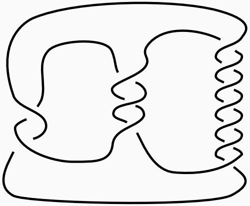
L-spaces

Fact: For a rational homology sphere M, $\dim \widehat{HF}(M) \geq |H_1(M, \mathbb{Z})|$.

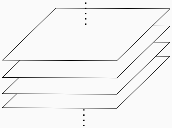
Definition: M is an L-space if $\dim \widehat{HF}(M) = |H_1(M, \mathbb{Z})|$.

Examples:

- S^3;
- lens spaces;
- M with finite fundamental group;
- Weeks manifold;
- r-surgery on $P(-2, 3, 7)$, for all $r \in [9, \infty]$.



Definition: A foliation $\mathcal{F}$ on M^3 is a decomposition of M into injectively immersed surfaces (leaves) such that

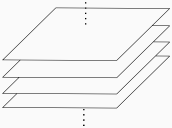
$(M, \mathcal{F}) \cong_{loc}$ 

$\mathbb{R}^2 \times \mathbb{R}$

Foliations

Definition: A foliation $\mathcal{F}$ on M^3 is a decomposition of M into injectively immersed surfaces (leaves) such that

$$(M, \mathcal{F}) \cong_{loc}$$

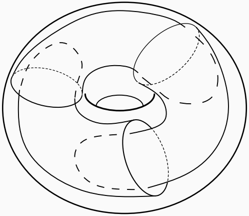


$\mathbb{R}^2 \times \mathbb{R}$

Theorem (Lickorish): Every closed oriented M^3 supports a foliation.

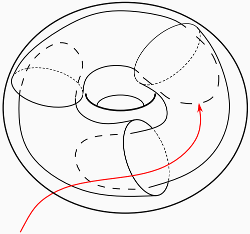
From Reebless to taut

The foliations constructed in Lickorish theorem contain *Reeb components*.



From Reebless to taut

The foliations constructed in Lickorish theorem contain *Reeb components*.

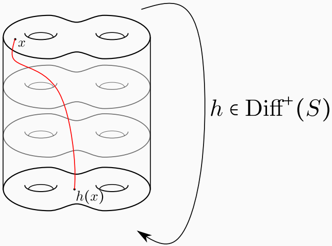


Taut foliations

Definition: A foliation $\mathcal{F}$ is *taut* if every leaf intersects a closed transversal.

Definition: A foliation $\mathcal{F}$ is *taut* if every leaf intersects a closed transversal.

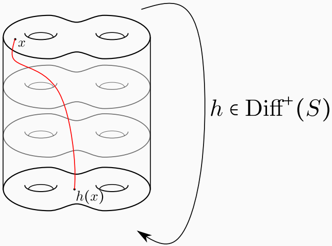
Example:



Taut foliations

Definition: A foliation $\mathcal{F}$ is *taut* if every leaf intersects a closed transversal.

Example:



Theorem (Novikov '65, Palmeira '78): If $M \neq S^2 \times S^1$ supports a coorientable taut foliation $\mathcal{F}$, then the leaves of $\mathcal{F}$ are incompressible and $\widetilde{M} \cong \mathbb{R}^3$.

Corollary: S^3, lens spaces and M with finite fundamental group *do not* support coorientable taut foliations.

The L-space conjecture

Corollary: S^3, lens spaces and M with finite fundamental group *do not* support coorientable taut foliations.

Theorem (Ozsváth-Szabó '04): If M supports a coorientable taut foliation, then M is not an L-space.

The L-space conjecture

Corollary: S^3, lens spaces and M with finite fundamental group *do not* support coorientable taut foliations.

Theorem (Ozsváth-Szabó '04): If M supports a coorientable taut foliation, then M is not an L-space.

L-space conjecture (Juhász '15):

Let M be an irreducible $\mathbb{Q}HS^3$. The following are equivalent:

(1) M supports a coorientable taut foliation (CTF);

(2) M is not an L-space (NLS);

The L-space conjecture

Corollary: S^3, lens spaces and M with finite fundamental group *do not* support coorientable taut foliations.

Theorem (Ozsváth-Szabó '04): If M supports a coorientable taut foliation, then M is not an L-space.

L-space conjecture (Juhász '15):

Let M be an irreducible $\mathbb{Q}HS^3$. The following are equivalent:

(1) M supports a coorientable taut foliation (CTF);

(2) M is not an L-space (NLS);

Theorem: The conjecture is true for graph manifolds.

The conjecture and two-bridge links

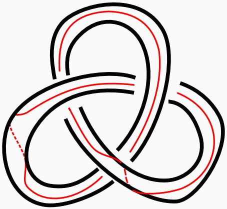
Dehn Surgery

K knot in S^3 and $\gamma \subset \partial\nu K$
unoriented essential simple closed curve $\rightsquigarrow$ $S^3_\gamma(K)$ surgery on K along γ

Dehn Surgery

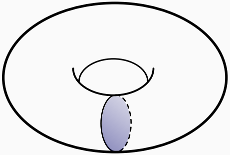
K knot in S^3 and $\gamma \subset \partial\nu K$
unoriented essential simple closed curve $\rightsquigarrow$ $S^3_\gamma(K)$ surgery on K along γ

$S^3 \setminus$

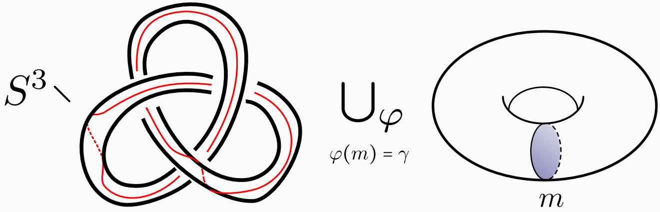


$\cup_\varphi$

$\varphi(m) = \gamma$



K knot in S^3 and $\gamma \subset \partial\nu K$ unoriented essential simple closed curve $\rightsquigarrow$ $S^3_\gamma(K)$ surgery on K along γ

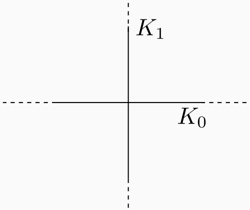


Theorem (Lickorish, Wallace): Every closed orientable M^3 can be obtained as surgery on some link in S^3.

$$
\begin{tikzcd}
\{\text{slopes on } K \text{ up to isotopy}\} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \frac{p}{q} \in \mathbb{Q} \cup \{\infty\} \arrow[dl, leftrightarrow] \\
\{\pm(p\mu + q\lambda) \in H_1(\partial\nu K, \mathbb{Z}) \mid (p,q) = 1\} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\{\text{slopes on } K \text{ up to isotopy}\} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \frac{p}{q} \in \mathbb{Q} \cup \{\infty\} \\
\{\pm(p\mu + q\lambda) \in H_1(\partial \nu K, \mathbb{Z}) \mid (p,q) = 1\} \arrow[ur, leftrightarrow] &
\end{tikzcd}
$$

K knot $\rightsquigarrow$

$L = K_0 \sqcup K_1$ link $\rightsquigarrow$ 

The conjecture and surgeries on knots

Theorem: If K is a nontrivial *L-space knot* (i.e. $S^3_r(K)$ is an L-space for some positive $r \in \mathbb{Q}$) then:

- K is fibered [Ghiggini '08, Ni '07];

The conjecture and surgeries on knots

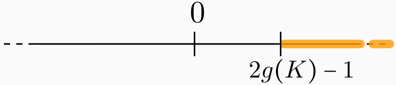
Theorem: If K is a nontrivial *L-space knot* (i.e. $S^3_r(K)$ is an L-space for some positive $r \in \mathbb{Q}$) then:

- K is fibered [Ghiggini '08, Ni '07];
- K is strongly quasipositive [Hedden '10];

The conjecture and surgeries on knots

Theorem: If K is a nontrivial *L-space knot* (i.e. $S^3_r(K)$ is an L-space for some positive $r \in \mathbb{Q}$) then:

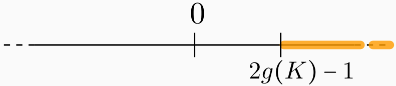
- K is fibered [Ghiggini '08, Ni '07];
- K is strongly quasipositive [Hedden '10];
- $S^3_t(K)$ is an L-space iff $t \in [2g(K) - 1, \infty]$ [KMOS '07].



THE CONJECTURE AND SURGERIES ON KNOTS

Theorem: If K is a nontrivial *L-space knot* (i.e. $S^3_r(K)$ is an L-space for some positive $r \in \mathbb{Q}$) then:

- K is fibered [Ghiggini '08, Ni '07];
- K is strongly quasipositive [Hedden '10];
- $S^3_t(K)$ is an L-space iff $t \in [2g(K) - 1, \infty]$ [KMOS '07].



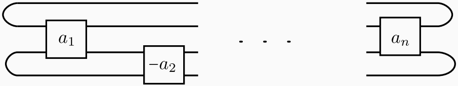
Roberts '01: The conjecture holds for all surgeries on the trefoil knot and the figure eight knot.

Krishna '20: The conjecture holds for all surgeries on an infinite family of hyperbolic L-space knots.

The main theorem

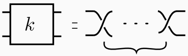
Theorem A (S. '23):

Let L be a fibered hyperbolic two-bridge link and let M be a manifold obtained as Dehn surgery on L. Then M admits a coorientable taut foliation if and only if M is not an L-space.



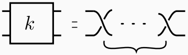
$$|a_i| = 2\ \forall i$$

k positive



k left half-twists

k negative



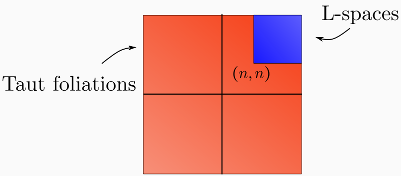
$|k|$ right half-twists

$$(a_1, \dots, a_n) \neq \pm(2, -2, 2, \dots, -2, 2)$$

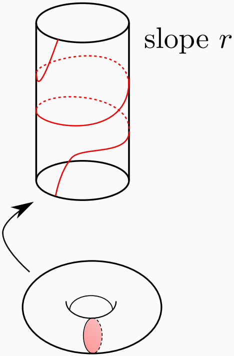
A Classification result

Theorem B (S. '23):

If a fibered hyperbolic two-bridge link L has a (finite) surgery that is an L-space, then L is isotopic, as unoriented link, to one of the links $\{L_n\}_{n\geq 1}$ or their mirrors.



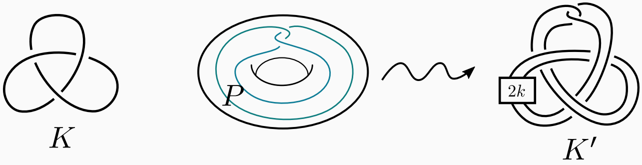
Remark: The foliation on the manifold $S^3_{r,s}(L)$ is obtained by extending a taut foliation on the exterior of L intersecting the two boundary components in parallel curves of slope r and s, respectively.



Application to Whitehead doubles

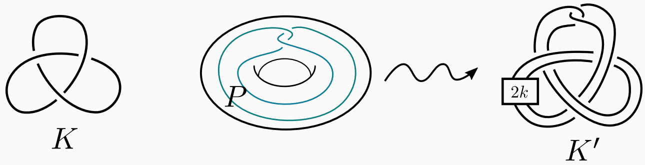
Whitehead doubles

Definition: Let $P \subset \mathbb{D}^2 \times S^1$ be the Whitehead pattern and let $\Phi : \mathbb{D}^2 \times S^1 \to \nu K$ be an orientation preserving diffeomorphism, where K is a knot in S^3. The knot $K' = \Phi(P)$ is a *Whitehead double* of K.



Whitehead doubles

Definition: Let $P \subset \mathbb{D}^2 \times S^1$ be the Whitehead pattern and let $\Phi : \mathbb{D}^2 \times S^1 \to \nu K$ be an orientation preserving diffeomorphism, where K is a knot in S^3. The knot $K' = \Phi(P)$ is a *Whitehead double* of K.



Fact: Whitehead doubles of nontrivial knots do not have (nontrivial) L-space surgeries.

Taut foliations and Whitehead doubles

Theorem C (S. '23):

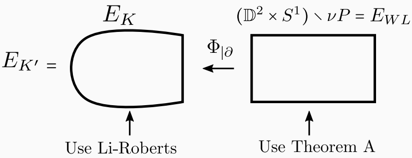
Let K be a nontrivial knot and let K' be a Whitehead double of K. Then all nontrivial surgeries on K' support a coorientable taut foliation.

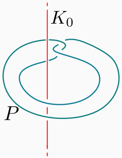
Taut foliations and Whitehead doubles

Theorem C (S. ’23):

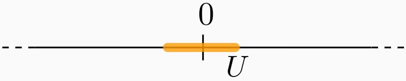
Let K be a nontrivial knot and let K' be a Whitehead double of K. Then all nontrivial surgeries on K' support a coorientable taut foliation.

Sketch of the proof.

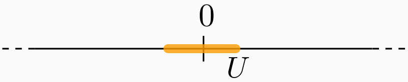


$\mathbb{D}^2 \times S^1 \smallsetminus \nu P = S^3 \smallsetminus \nu$ 

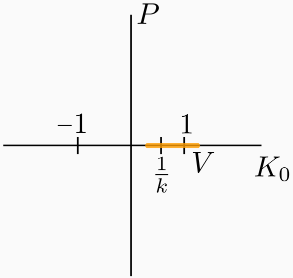
Li-Roberts '14: $\exists U$ neighbourhood of 0 such that all slopes on K in U are realised by a CTF in the exterior of K.



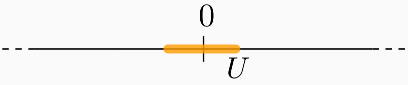
Li-Roberts ’14: $\exists U$ neighbourhood of 0 such that all slopes on K in U are realised by a CTF in the exterior of K.



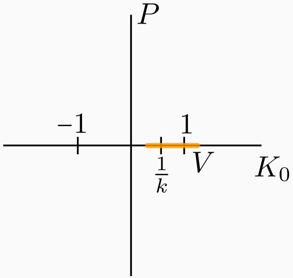
The map $\Phi_{|\partial}$ identifies U with a neighbourhood V of $\frac{1}{k} \in \mathbb{Q} \cup \{\infty\}$ ={Slopes on K_0}, for some $k \in \mathbb{Z}$.

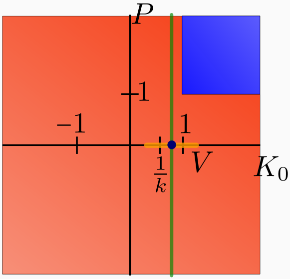


Li-Roberts '14: $\exists U$ neighbourhood of 0 such that all slopes on K in U are realised by a CTF in the exterior of K.



The map $\Phi_{|\partial}$ identifies U with a neighbourhood V of $\frac{1}{k} \in \mathbb{Q} \cup \{\infty\} =$ {Slopes on K_0}, for some $k \in \mathbb{Z}$.





For the general statement of the L-space conjecture, for more details, and more pictures… see you at the poster session!