Constructions of branched coverings in dimension four

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Notion of branched covering

$$p\colon X^n \to Y^n$$

branched covering $\overset{\text{def}}{\iff}$ p is proper open finite PL map of compact n-manifolds.

- $B_p \subset Y$ branch set of p: max subspace over which p fails to be a loc. homeo (codim-2 subcomplex).

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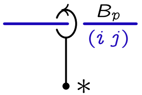
- p uniquely determined by (Y, B_p, ω_p), up to homeo.
- p simple $\overset{\text{def}}{\iff}$ $\{\text{meridians of } B_p\} \xrightarrow{\omega_p} \{\text{transpositions}\}$.



Theorem (Hilden-Hirsch-Montesinos 1974)

Any closed oriented M^3 is simple 3-fold b.c.

$$p \colon M^3 \longrightarrow S^3$$

branched over a link $B_p \subset S^3$.

Connections with other structures

- Open book decompositions (Myers 1978).
- Crystallizations (Ferri 1979; Casali, Cavicchioli, Grasselli 1989).
- Contact structures (Gonzalo 1987; Giroux 2000).

Theorem (Piergallini 1995)

Any closed oriented PL M^4 is a simple 4-fold b.c.

$$p \colon M^4 \longrightarrow S^4$$

branched over a loc. flat immersed surface $B_p \subset S^4$.

(B_p embedded and $d = 5$, Iori & Piergallini 2002).

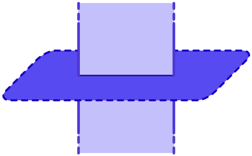
Remark

If M^4 admits handle decomp. with no 1-handles can take $d = 3$ and B_p loc. flat except at a conical singularity *(Blair, Cahn, Kjuchukova & Meier*, to appear).

Ribbon surface

$$(S, \partial S) \subset (B^4, S^3)$$

push in of immersed ribbon surface $S' \in S^3$ having only ribbon intersections as possible singularities.



Remark

$\partial S \subset S^3$ knot or link.

Simple branched covering

$$p\colon M^3 \xrightarrow{\text{b.c.}} S^3$$

branched over link that can be extended to simple b. c.

$$q\colon W^4 \to B^4$$

branched over ribbon surface, $\partial W = M$, $\partial q = q_{|\partial W} = p$, $\partial B_q = B_p$.

Theorem (Montesinos 1978)

Any compact oriented 4-dim 2-handlebody

$$W^4 = H^0 \cup_m H^1 \cup_n H^2$$

is a simple 3-fold b.c.

$$p \colon W^4 \longrightarrow B^4$$

branched over a ribbon surface $B_p \subset B^4$.

Remark

By definition $p_{|\partial W} \colon \partial W \to S^3$ is ribbon fillable.

Cobordism Lemma (Piergallini-Z. 2019)

$\forall\, p_0,\ p_1 : M^3 \to S^3$ *d-fold ribbon fillable branched covers,* $\forall\, d \geqslant 5$
$\Rightarrow\ \exists$ *simple covering*

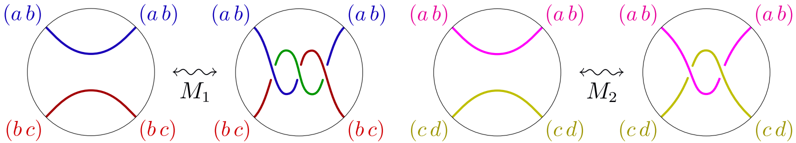
$$q\colon M^3 \times [0,1] \longrightarrow S^3 \times [0,1]$$

branched over a loc. flat surface, s. t.

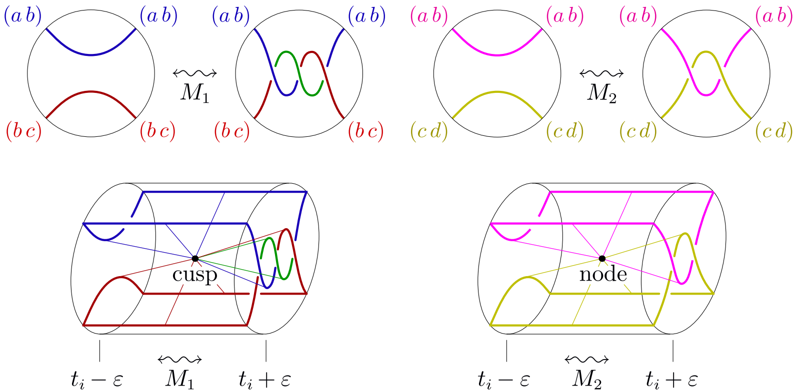
$$p_0 = q_{|M\times 0} \quad \textit{and} \quad p_1 = q_{|M\times 1}.$$

$p_0, p_1 : M^3 \to S^3$ can be related by finitely many Montesinos moves M_1 & M_2 and isotopy.

(Conjectured by Montesinos 1985,
proved by Bobtcheva & Piergallini 2012)



Sketch of proof



Start with

$$p_0 \times \mathrm{id} \colon M^3 \times [0,1] \longrightarrow S^3 \times [0,1]$$

add cusps for M_1's, nodes for M_2's, movies for isotopy.

Removing cusps *(Piergallini 1995)* and nodes *(Iori & Piergallini 2002)* yields the desired cobordism branched covering (here we use ribbon fillable).

Remark

By allowing nodes as singularities of B_q one can take $d = 4$.

Theorem (Piergallini-Z. 2019)

$\forall$ cpt connected orient. PL W^4 with connected ∂W, $\forall\, d \geqslant 5$, $\exists$ simple d-fold branched covering

$$p : W^4 \longrightarrow B^4$$

with $B_p \subset B^4$ loc. flat surface.
We can assume $\partial p : \partial W \to S^3$ to coincide with any given ribbon fillable branched covering.

Theorem (Piergallini-Z. 2021)

For any closed oriented PL M^4

$$\exists\ p\colon M^4 \xrightarrow{b.c.} \mathbb{CP}^2 \iff b_2^+(M^4) \geqslant 1.$$

We can assume $B_p \subset \mathbb{CP}^2$ loc. flat surf. and $d \leqslant 9$.

Remark

$b_2^\pm = (b_2 \pm \sigma)/2 \in \mathbb{N}$.

If $\exists$ d-fold b.c.

$$p\colon M^4 \longrightarrow \mathbb{CP}^2$$

isotope $B_p \subset \mathbb{CP}^2$ to be transversal to $\mathbb{CP}^1$.

$$F = p^{-1}(\mathbb{CP}^1) \subset M^4$$

$$F \cdot F = d \neq 0 \ \Rightarrow [F] \neq 0 \text{ in } H_2(M)/\mathrm{Tor}.$$

$\exists$ orient. connect. surf. $F \subset M^4$ s. t. $F \cdot F \geqslant 1$.
Can assume $d = F \cdot F \geqslant 5$ (otherwise take $F' \in 3[F]$).

$$\begin{tikzcd}
M^4 = & X \arrow[d, "q_2"', "\text{b. c.}"] & \cup_\partial & N_M(F) \arrow[r, "\pi' = q^*(\pi)", "B^2\text{–bundle}"'] \arrow[d, "q_1"', "\substack{e = d \\ d:1 \\ e = 1}"] & F \arrow[d, "d:1"', "q \quad \text{b. c.}"] \\
\mathbb{CP}^2 = & B^4 & \cup_\partial & N_{\mathbb{CP}^2}(L) \arrow[r, "B^2\text{–bundle}", "\pi"'] & L = \mathbb{CP}^1
\end{tikzcd}$$

Glue by Cobordism Lemma $\qquad$ N = Tub Nbrd

$q : F \to \mathbb{CP}^1$ branched over $g(F) + d - 1$ pairs of points $\Rightarrow \partial q_1$ branched over pairs of Hopf links bounding Hopf bands $\Rightarrow \partial q_1$ ribbon fillable.

$\bullet\,\bullet \;\cdots\; \bullet\,\bullet\;\bullet\,\bullet \;\cdots\; \bullet\,\bullet$

$(1\ 2)$ $\quad$ $(2\ 3)$ $\quad$ $(d-1\ \ d)$

Theorem (Piergallini-Z. 2019)

For any closed oriented PL M^4

$$\exists\ p : M^4 \xrightarrow{b.c.} N^4$$

with

- $N = \#_m \mathbb{CP} \#_n \overline{\mathbb{CP}} \iff b_2^+(M) \geqslant m$ *and* $b_2^-(M) \geqslant n$.

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- $N = \#_n(S^2 \times S^2) \iff b_2^+(M) \geqslant n$ *and* $b_2^-(M) \geqslant n$.

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- $N = \#_n(S^2 \times S^2) \iff b_2^+(M) \geqslant n$ *and* $b_2^-(M) \geqslant n$.
- $N = \#_n(S^3 \times S^1) \iff \pi_1(M) \twoheadrightarrow F_n$.

$b_2^{\pm}(T^4) = 3 \Rightarrow \exists\ T^4 \xrightarrow{\text{b.c.}} \#_n(S^2 \times S^2)$ for $n = 1, 2, 3$.

Case $n = 1$ obvious: $T^4 = T^2 \times T^2 \xrightarrow{p \times p} S^2 \times S^2$.

Case $n = 2$ proved by Rickman (2006).

Case $n = 3$ not previously known.

Definition

M^m is quasiregularly elliptic if $\exists\ f : \mathbb{R}^m \to M^m$ s. t.

$$\|D f\|^m \leqslant K \det D f .$$

Corollary

$\#_n(S^2 \times S^2)$ *quasiregularly elliptic for* $n \leqslant 3$

(False for all $n \geqslant 4$, Prywes 2019).

Proof.

$\mathbb{R}^4 \xrightarrow{\text{cover}} T^4 \xrightarrow{\text{b.c.}} \#_n(S^2 \times S^2)$. $\square$

Theorem (Piergallini-Z. 2019)

For any closed conn. orient. loc. flat pair (M^4, N^2) *s. t.*

$$d = |N \cdot N| \geqslant 4$$

there is a simple d*-fold covering*

$$p\colon (M^4, N^2) \longrightarrow (\pm\mathbb{CP}^2, \mathbb{CP}^1)$$

branched over a (nodal) surface transversal to $\mathbb{CP}^1$*, with* $\pm = \mathrm{sgn}(N \cdot N)$.

Theorem (Piergallini-Z. 2019)

For any closed conn. orient. loc. flat pair (M^4, N^2) s. t.

$$N \cdot N = 0$$

and for any $d \geqslant 4$ there is a simple d-fold covering

$$p\colon (M^4, N^2) \longrightarrow (S^4, S^2)$$

branched over a (nodal) surface transversal to S^2.

Special case

$p\colon (S^4, N^2) \xrightarrow{\text{b. c.}} (S^4, S^2)$ for every 2-knot $N^2 \subset S^4$.

Theorem (Piergallini-Z. 2019)

For any closed conn. orient. pair (M^4, N^3) there is a d-fold simple b.c.

$$p\colon (M^4, N^3) \longrightarrow (S^4, S^3).$$

The branch set can be taken an immersed (embedded for $d \geqslant 5$) surface transversal to $S^3 \subset S^4$.

Remark

If N^3 disconnects M^4 we can assume $N^3 = p^{-1}(S^3)$.

A remarkable case is that of

$$S^3 \cong \Sigma^3 \underset{\mathrm{PL}}{\subset} S^4.$$

4D PL Schoenflies Conjecture

$$(S^4, \Sigma^3) \underset{\mathrm{PL}}{\cong} (S^4, S^3).$$

Remark

The topological version is well-known to be true (Brown 1960)

$$(S^4, \Sigma^3) \underset{C^0}{\cong} (S^4, S^3).$$

Corollary (Piergallini-Z. 2019)

$\forall\, S^3 \cong \Sigma^3 \underset{PL}{\subset} S^4$ and $\forall\, d \geqslant 4$ there is a d-fold simple covering

$$p\colon (S^4, \Sigma^3) \longrightarrow (S^4, S^3)$$

branched over a (nodal) surface transversal to S^3.

- PL $=$ Diff in dim 4 $\Rightarrow$ similar results in the C^∞-category.
- Closed symplectic 4-manifolds are coverings of $\mathbb{CP}^2$ branched over symplectic surface with cusps and nodes *(Auroux 2000).*
- Nodes become removable by assuming $d \geqslant 5$
  $\Rightarrow$ non-singular branch surface.